sound oscillations has to be considered in the evaluation of the thermodynamic functions of high-temperature gases only at high densities.

Gases with a considerable acoustic noise background are encountered in various high-temperature engineering systems, such as gas turbines, jet engines, rocket exhausts, etc. The theory presented permits calculation of the free energy ΔF of the acoustic degrees of freedom in such systems, provided that the acoustic noise is in thermal equilibrium. Considerably larger free-energy contributions are to be expected under nonequilibrium conditions, particularly if the acoustic fluctuations exhibit intensity levels corresponding to turbulence.

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Higher Order Sensitivities in Structural Systems

H. Van Belle*
BN Spoorwegmaterieel en Metaalconstructies, Brussels, Belgium

Introduction

ALTHOUGH several papers on sensitivity analysis have been published in the last few years, there has been little interest in higher order sensitivities. Higher order sensitivities are especially important if the relation between finite difference and differential sensitivities or an expression for the direct calculation of difference sensitivities is not available. This is, as far as we know, the case for difference sensitivities of natural frequencies and mode shapes. In this Note a flexibility method for the calculation of higher order sensitivities of flexibilities, structural eigenvalues, and flexibility modes is introduced. If the first-order and a number of higher order sensitivities are available, an approximation of the difference sensitivities based on a Taylor series becomes possible.

Most of the papers dealing with sensitivity analysis concern only first-order differential sensitivities of eigenvalues and eigenvectors. These derivatives are used to approximate the finite difference or large-change sensitivities. The authors use the stiffness matrix formulation and assume that the stiffness matrices are assemblages of matrices of the component elements.[1-5]

Since finite difference sensitivities are in general nonlinear functions of the structural parameters, the derivatives cannot yield good approximations if large parameter changes are considered. With the help of truncated Taylor series and a number of higher order sensitivities a better approximation is possible. An exact and simple matrix equation for the calculation of the large-change sensitivity of eigenvalues and eigenvectors is still not available.

The widely used stiffness methods are of limited utility in the important field of experimental modal analysis. In this case data for partial flexibility matrices are measured by Fourier analyzers. Curve-fitting techniques are used to extract the eigenvalues and flexibility modes (or residues). Only a limited number of important modes is considered.

The assumption that the static stiffness matrix, which appears in the system equation, consists of a superposition of several element matrices is not always valid, even in computer-aided design. If the mass is concentrated in the nodes and mass moments of inertia are not taken into account, the full stiffness matrix must be reduced to a "pseudostiffness" matrix. When the rotational degrees of freedom are eliminated, the linear characteristic of the stiffness matrix with respect to the element stiffnesses is lost.

In both cases (modal analysis and computer-aided design with a simple lumped-mass model) the classical methods are not very useful. Some methods developed in system and electrical network theory can be generalized and transferred successfully to mechanical structural sensitivity analysis.[6]

Flexibility Sensitivities

Consider the structure represented in Fig. 1. This linear structure consists of two connected parts: a nonvariable substructure and a variable element. Suppose that the variable element is characterized by a stiffness matrix $|K_{ll}|$ which is a function of at least one parameter R_m. For the entire structure the flexibility matrix $|S'|$ with partial matrices $|S'_{rq}|$ will be used as a model.

The difference sensitivity of a partial flexibility matrix $|S'_{rq}|$ for a (large) element parameter change ΔR_m can be written as

$$\frac{\Delta |S'_{rq}|}{\Delta R_m} = -|S'_{rl}| \frac{\Delta |K_{ll}|}{\Delta R_m} [\,|I| + |S'_{ll}|\,\Delta |K_{ll}|\,]^{-1}\, |S'_{lq}| \qquad (1)$$

The evidence is given in Ref. 6. Differential sensitivities become equal to difference sensitivities if infinitesimal changes are introduced. A limit transformation results immediately in

$$\frac{\partial |S'_{rq}|}{\partial R_m} = -|S'_{rl}| \frac{\partial |K_{ll}|}{\partial R_m} |S'_{lq}| \qquad (2)$$

In the special case that

$$\frac{\Delta |K_{ll}|}{\Delta R_m} = \frac{\partial |K_{ll}|}{\partial R_m} \qquad (3)$$

the relation between difference and differential sensitivities for some partial matrices of the flexibility matrix can be

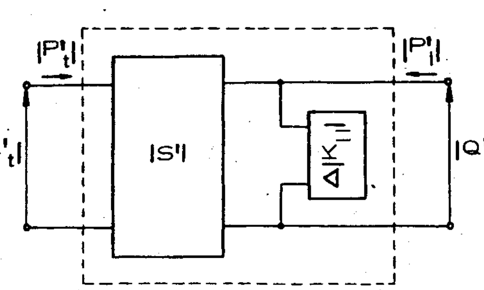

Fig. 1 Generalized *n* port representation of a modified structure.

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*Assistant Head, Corporate Strategic Planning.

expressed as

$$\frac{\Delta |S'_{rq}|}{\Delta R_m} = \frac{\partial |S'_{rq}|}{\partial R_m} [\, |S'_{1q}|^t |S'_{1q}| \,]^{-1} |S'_{1q}|^t$$
$$\times [\, |1| + |S'_{11}| \Delta |K_{11}| \,]^{-1} |S'_{1q}| \tag{4}$$

It is possible to prove this relation with the help of the (left) pseudoinverse

$$|S'_{1q}|^- = [\, |S'_{1q}|^t |S'_{1q}| \,]^{-1} |S'_{1q}|^t \tag{5}$$

This pseudoinverse exists only if the number of rows of $|S'_{1q}|$ is greater than or equal to the number of columns.

The difference sensitivity may be expanded in a Taylor series as

$$\frac{\Delta |S'_{rq}|}{\Delta R_m} = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k |S'_{rq}|}{\partial R_m^k} (\Delta R_m)^{k-1} \tag{6}$$

If the matrix inversion is substituted by a Maclaurin series, Eq. (1) becomes:

$$\frac{\Delta |S'_{rq}|}{\Delta R_m} = - |S'_{r1}| \frac{\Delta |K_{11}|}{\Delta R_m} \left\{ \sum_{k=1}^{\infty} (-1)^{k-1} \right.$$
$$\left. \times \left[|S'_{11}| \frac{\Delta |K_{11}|}{\Delta R_m} \right]^{k-1} (\Delta R_m)^{k-1} \right\} |S'_{1q}| \tag{7}$$

It follows that the (first and) higher order sensitivities are given by

$$\frac{\partial^k |S'_{rq}|}{\partial R_m^k} = (-1)^k k! \, |S'_{r1}| \frac{\partial |K_{11}|}{\partial R_m} \left[|S'_{11}| \frac{\partial |K_{11}|}{\partial R_m} \right]^{k-1} |S'_{1q}| \tag{8}$$

if Eq. (3) holds. In this case $|K_{11}|$ is a linear function of the parameter R_m so that the difference sensitivity $\Delta |K_{11}| / \Delta R_m$ no longer depends on ΔR_m.

The relation between the first and higher order sensitivities of a partial matrix of the flexibility matrix becomes equal to

$$\frac{\partial^k |S'_{rq}|}{\partial R_m^k} = (-1)^{k-1} k! \frac{\partial |S'_{rq}|}{\partial R_m} [\, |S'_{1q}|^t |S'_{1q}| \,]^{-1} |S'_{1q}|^t$$
$$\times \left[|S'_{11}| \frac{\partial |K_{11}|}{\partial R_m} \right]^{k-1} |S'_{1q}| \tag{9}$$

in case that the pseudoinverse of Eq. (5) exists.

Note that Eqs. (4) and (9) are not only applicable to partial matrices $|S'_{rq}|$ but also to single elements S'_{rq} of the flexibility matrix. We can conclude that a flexibility matrix contains the data for sensitivity calculation.

Sensitivities of Structural Eigenvalues and Flexibility Modes

Classical methods of sensitivity analysis of natural frequencies and mode shapes are based on the system equations and assume stiffness, mass, and damping matrices which consist of a superposition of several element matrices.

A method for the calculation of first-order differential sensitivities of natural frequencies and flexibility modes for undamped structures, based on a transfer function expressed with natural frequencies and flexibility modes, was introduced in Ref. 7. This method imposes no conditions on the stiffness, mass, and damping matrices and can easily be generalized for viscous damped structures and higher order sensitivities.

The dynamic behavior of a linear structure with n degrees of freedom may be described by

$$|S'_{rq}| = \sum_{i=1}^{2n} \frac{|S'_{i,rq}|}{p - \lambda_i} \tag{10}$$

where $|S'_{i,rq}|$ = the ith (complex) flexibility mode or residue matrix, p = the Laplace operator, and λ_i = the ith eigenvalue or pole.

Equation (10) corresponds to a transfer function with different poles expanded in partial fractions. The flexibility modes are functions of the corresponding (complex) mode shapes. Structural eigenvalues and flexibility modes appear in complex conjugate pairs.

There are two ways to calculate the differential sensitivity of a partial flexibility matrix:

1) Through a direct partial differentation of Eq. (10)

$$\frac{\partial |S'_{rq}|}{\partial R_m} = \sum_{i=1}^{2n} \left[\frac{1}{p - \lambda_i} \frac{\partial |S'_{i,rq}|}{\partial R_m} + \frac{|S'_{i,rq}|}{(p - \lambda_i)^2} \frac{\partial \lambda_i}{\partial R_m} \right] \tag{11}$$

2) Applying Eq. (2) for flexibilities expressed as in Eq. (10)

$$\frac{\partial |S'_{rq}|}{\partial R_m} = - \left[\sum_{i=1}^{2n} \frac{|S'_{i,r1}|}{p - \lambda_i} \right] \frac{\partial |K_{11}|}{\partial R_m} \left[\sum_{i=1}^{2n} \frac{|S'_{i,1q}|}{p - \lambda_i} \right] \tag{12}$$

This equation can also be written as

$$\frac{\partial |S'_{rq}|}{\partial R_m} = - \sum_{i=1}^{2n} \frac{|S'_{i,r1}|}{p - \lambda_i} \frac{\partial |K_{11}|}{\partial R_m} \frac{|S'_{i,1q}|}{p - \lambda_i}$$
$$- \sum_{i=1}^{2n} \sum_{\substack{j=1 \\ j \neq i}}^{2n} \frac{|S'_{i,r1}|}{p - \lambda_i} \frac{\partial |K_{11}|}{\partial R_m} \frac{|S'_{j,r1}|}{p - \lambda_j} \tag{13}$$

An expansion in partial fraction gives

$$\frac{\partial |S'_{rq}|}{\partial R_m} = - \sum_{i=1}^{2n} \left[\frac{|A'_i|}{(p - \lambda_i)^2} + \frac{|B'_i|}{p - \lambda_i} \right]$$
$$- \sum_{i=1}^{2n} \sum_{\substack{j=1 \\ j \neq i}}^{2n} \left[\frac{|C'_{ij}|}{p - \lambda_i} + \frac{|D'_{ji}|}{p - \lambda_j} \right] \tag{14}$$

with

$$|A'_i| = |S'_{i,r1}| \left[\frac{\partial |K_{11}|}{\partial R_m} \right]_{p=\lambda_i} |S'_{i,1q}| \tag{15}$$

$$|B'_i| = |S'_{i,r1}| \left\{ \frac{d}{dp} \left[\frac{\partial |K_{11}|}{\partial R_m} \right] \right\}_{p=\lambda_i} |S'_{i,1q}| \tag{16}$$

$$|C'_{ij}| = |S'_{i,r1}| \left[\frac{\partial |K_{11}|}{\partial R_m} \right]_{p=\lambda_i} \frac{|S'_{j,1q}|}{\lambda_i - \lambda_j} \tag{17}$$

$$|D'_{ji}| = \frac{|S'_{i,r1}|}{\lambda_j - \lambda_i} \left[\frac{\partial |K_{11}|}{\partial R_m} \right]_{p=\lambda_j} |S'_{j,1q}| \tag{18}$$

Identification of Eqs. (11) and (14) yields formulas for the calculation of the differential sensitivities of eigenvalues and flexibility modes

$$\frac{\partial \lambda_i}{\partial R_m} |1| = - |S'_{i,rq}|^{-1} |A'_i| \tag{19}$$

$$\frac{\partial |S'_{i,rq}|}{\partial R_m} = - |B'_i| - \sum_{\substack{j=1 \\ j \neq i}}^{2n} [\, |C'_{ij}| + |D'_{ij}| \,] \tag{20}$$

in which $|S'_{i,rq}|$ is a square, nonsingular matrix.

This technique can also be applied for higher order sensitivities. For instance, the basic equations for the determination of the second-order (differential) sensitivities are

$$\frac{\partial^2 |S'_{rq}|}{\partial R_m^2} = \sum_{i=1}^{2n} \left[\frac{1}{p - \lambda_i} \frac{\partial^2 |S'_{i,rq}|}{\partial R_m^2} + \frac{2}{(p - \lambda_i)^2} \frac{\partial \lambda_i}{\partial R_m} \frac{\partial |S'_{i,rq}|}{\partial R_m} \right.$$
$$\left. + \frac{|S'_{i,rq}|}{(p - \lambda_i)^2} \frac{\partial^2 \lambda_i}{\partial R_m^2} + 2 \frac{|S'_{i,rq}|}{(p - \lambda_i)^3} \left(\frac{\partial \lambda_i}{\partial R_m} \right)^2 \right] \tag{21}$$

$$\frac{\partial^2 |S'_{rq}|}{\partial R_m^2} = 2\left[\sum_{i=1}^{2n} \frac{|S'_{i,r1}|}{p-\lambda_i}\right]\frac{\partial |K_{11}|}{\partial R_m}\left[\sum_{i=1}^{2n} \frac{|S'_{i,11}|}{p-\lambda_i}\right]\frac{\partial |K_{11}|}{\partial R_m} \times \left[\sum_{i=1}^{2n} \frac{|S'_{i,1q}|}{p-\lambda_i}\right] \tag{22}$$

After identification we find the eigenvalue second-order differential sensitivity

$$\frac{\partial^2 \lambda_i}{\partial R_m^2}|I| = 2|S'_{i,rq}|^{-1}\left\{|F'_i| + \sum_{\substack{j=1\\ j\neq i}}^{2n}\left[|G'_{ij}| + |H'_{ij}| + |K'_{ij}|\right] - \frac{\partial \lambda_i}{\partial R_m}\frac{\partial |S'_{i,rq}|}{\partial R_m}\right\} \tag{23}$$

with

$$|F'_i| = |S'_{i,r1}|\left\{\frac{d}{dp}\left[\frac{\partial |K_{11}|}{\partial R_m}\right]\right\}_{p=\lambda_i}|S'_{i,11}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,1q}| + |S'_{i,r1}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,11}|\left\{\frac{d}{dp}\left[\frac{\partial |K_{11}|}{\partial R_m}\right]\right\}_{p=\lambda_i}|S'_{i,1q}| \tag{24}$$

$$|G'_{ij}| = \frac{|S'_{j,r1}|}{\lambda_i-\lambda_j}\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,11}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,1q}| \tag{25}$$

$$|H'_{ij}| = |S'_{i,r1}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}\frac{|S'_{j,11}|}{\lambda_i-\lambda_j}\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,1q}| \tag{26}$$

$$|K'_{ij}| = |S'_{i,r1}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,11}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}\frac{|S'_{j,1q}|}{\lambda_i-\lambda_j} \tag{27}$$

The identification procedure leads also to

$$\left(\frac{\partial \lambda_i}{\partial R_m}\right)^2 |I| = |S'_{i,rq}|^{-1}|E'_i| \tag{28}$$

with

$$|E'_i| = |S'_{i,r1}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,11}|\left[\frac{\partial |K_{11}|}{\partial R_m}\right]_{p=\lambda_i}|S'_{i,1q}| \tag{29}$$

Taking in account Eqs. (19) and (15) we get

$$|S'_{i,11}| = |S'_{i,1q}|\,|S'_{i,rq}|^{-1}|S'_{i,r1}| \tag{30}$$

This equation is well known in the field of modal analysis. It follows that

$$|F'_i| = |B'_i|\,|S'_{i,rq}|^{-1}|A'_i| + |A'_i|\,|S'_{i,rq}|^{-1}|B'_i| \tag{31}$$

$$|G'_{ij}| = |D'_{ij}|\,|S'_{i,rq}|^{-1}|A'_i| \tag{32}$$

$$|H'_{ij}| = (\lambda_i-\lambda_j)\,|C'_{ij}|\,|S'_{j,rq}|^{-1}|D'_{ij}| \tag{33}$$

$$|K'_{ij}| = |A'_i|\,|S'_{i,rq}|^{-1}|C'_{ij}| \tag{34}$$

The introduction of Eqs. (19), (20), and (31-34) in Eq. (23) finally results in

$$\frac{\partial^2 \lambda_i}{\partial R_m^2}|I| = 2|S'_{i,rq}|^{-1}\left\{|A'_i|\,|S'_{i,rq}|^{-1}|B'_i| + \sum_{\substack{j=1\\ j\neq i}}^{2n}\left[(\lambda_i-\lambda_j)|C'_{ij}|\,|S'_{j,rq}|^{-1}|D'_{ij}|\right]\right\} \tag{35}$$

A similar expression for the second-order (differential) sensitivity of the flexibility modes can be derived with the same procedure. Notice that the sensitivities of the flexibility mode matrices $|S'_{i,rq}|$, $|S'_{i,r1}|$, and $|S'_{i,1q}|$ are not used to calculate the higher order sensitivities in this method. This should be the case if Eqs. (19) and (20) were directly differentiated. The method can be applied only when all important eigenvalues and flexibility modes are known.

Conclusions

Higher order sensitivities are as mentioned especially important for the determination of the influence of large modifications on structural eigenvalues and flexibility modes. The described method provides formulas to calculate the first-order and some higher order sensitivities which make an approximation of the finite difference sensitivities with the help of a Taylor series possible. Applications for structural optimization as well in computer-aided design as experimental modal analysis are obvious.

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